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An Electromagnetic Interpretation of the Kerr–Vaidya Metric

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Abstract

It is shown that at large distances from a rotating mass, the radiation may be associated with an Einstein-Maxwell null field with a non-zero null current.

1. *Introduction*

It is known that the Schwarzschild metric may be put in the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2\,du\,dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

in the co-ordinates $x^\mu = (u, r, \theta, \phi)$, $\mu = 0, 1, 2, 3$. Vaidya's radiating metric (1953) is obtained by making m an arbitrary function of the co-ordinate u. The resulting gravitational field equations are then

$$R_{\alpha\beta} = \frac{2m'}{r^2} l_\alpha l_\beta \tag{1.1}$$

where $m' = [dm(u)/du]$ and l_α is the null vector (1,0,0,0).

Murenbeeld & Trollope (1970) have suggested a corresponding generalisation of the Kerr metric, represented by the metric tensor,

$$g_{\alpha\beta} = \begin{bmatrix} 1 - \dfrac{2mr}{\rho^2} & 1 & 0 & \dfrac{2ma}{\rho^2} r \sin^2\theta \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & -\rho^2 & 0 \\ \dfrac{2ma}{\rho^2} r\sin^2\theta & -a\sin^2\theta & 0 & -\sin^2\theta\left[\dfrac{2ma^2}{\rho^2} r\sin^2\theta + r^2 + a^2\right] \end{bmatrix} \tag{1.2}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$.

With m a function of u and a a constant, they obtain the gravitational field equations in the form

$$R_{\alpha\beta} = qw_\alpha w_\beta + w_\alpha a_\beta + w_\beta a_\alpha \tag{1.3}$$

where w_α is a null vector and $w_\alpha a^\alpha = 0$, and discuss the case $(a/m)^2 \ll 1$ for which (1.3) is approximately of the form (1.1).

The object of this paper is to provide an interpretation of the energy flux in the case considered by Murenbeeld and Trollope in terms of electromagnetic radiation. It is found that in the approximation considered by Murenbeeld and Trollope, the space-time defined by (1.2) admits an electromagnetic field with non-zero current density. The field differs from that associated with the Vaidya metric (Goodinson and Newing, 1971) in that the total rate of energy emission is constant and an extra term arises in the current density vector J^μ.

The general case $(a/m)^2 \not\ll 1$ is first considered, for which the non-zero components of the Ricci tensor obtained by Murenbeeld and Trollope are,

$$\left.\begin{aligned}
R_{00} &= 2m' r^2(r^2 + a^2)/\rho^6 + (m'' ra^2 \sin^2\theta)/\rho^4 \\
R_{03} &= R_{30} = -a\sin^2\theta[R_{00} + m'(r^2 - a^2\cos^2\theta)/\rho^4] \\
R_{33} &= a^2\sin^4\theta[R_{00} + 2m'(r^2 - a^2\cos^2\theta)/\rho^4] \\
R_{02} &= R_{20} = (2m' ra^2 \sin\theta\cos\theta)/\rho^4 \\
R_{32} &= R_{23} = -a\sin^2\theta \,.\, R_{02}
\end{aligned}\right\} \tag{1.4}$$

It may be noted that since co-ordinate singularities are to be avoided, the restriction $r > 2m$ must hold and it is found more convenient in later sections to work with the approximation $(a/r)^2 \ll 1$ rather than $(a/m)^2 \ll 1$.

2. *The Rainich Conditions*

With reference to Goodinson and Newing (1968) the conditions for an Einstein–Maxwell field include the requirements that

(i) $R_\alpha{}^\alpha = 0$ (2.1)

(ii) $R_{\alpha\beta} V^\alpha V^\beta < 0$ where V^α is any time-like vector (2.2)

(iii) $R_{\mu\alpha} R^{\alpha\beta} = \Lambda^2 \delta_\mu{}^\beta$ (2.3)

It is found at once that (2.1), (2.3) are not satisfied with (1.4) values of $R_{\alpha\beta}$ and so the Kerr–Vaidya field cannot be interpreted as arising purely from an electromagnetic field.

Consider now condition (2.2). With reference to Goodinson and Newing (1969) a tetrad of null vectors l_α, m_α, $\bar{m}_\alpha$, n_α can be constructed to describe the space-time, and any arbitrary vector will have an expansion in terms of these tetrad vectors.

Writing

$$V^\alpha = l^\alpha + dn^\alpha + bm^\alpha + \bar{b}\bar{m}^\alpha$$

where d is real, then if V^α is time-like, $d > b\bar{b}$. Computing $R_{\alpha\beta} V^\alpha V^\beta$ using (1.4) gives

$$R_{\alpha\beta} V^\alpha V^\beta = d^2 \left\{ \frac{2m' r^2}{\rho^6}(r^2 + a^2) + \frac{m'' ra^2 \sin^2\theta}{\rho^4} \right\} + \frac{2m' a \sin\theta}{\sqrt{(2)}.\rho^4}. d\{b(a\cos\theta - ir) + \bar{b}(a\cos\theta + ir)\}$$

which must be <0 in order to ensure a positive energy density. For convenience put $b = b_0 e^{i\beta}$, $d = b_0{}^2 \eta$ where η is subject to $\eta > 1$. Then for positive energy density,

$$b_0 \eta \left\{ \frac{2m' r^2}{\rho^6}(r^2 + a^2) + \frac{m'' ra^2 \sin^2\theta}{\rho^4} \right\} + \frac{4m' a \sin\theta}{\sqrt{(2)}.\rho^4}(a\cos\theta\cos\beta + r\sin\beta) < 0$$

for all β and η.

Re-writing the above as

$$b_0 \eta X + Y \sin(\alpha + \beta) < 0 \tag{2.4}$$

where

$$X = \frac{2m' r^2(r^2 + a^2)}{\rho^6} + \frac{m'' ra^2 \sin^2\theta}{e^4} = R_{00}$$

$$Y = \frac{4m' a \sin\theta}{\sqrt{(2)}\rho^3}, \quad \tan\alpha = \frac{a\cos\theta}{r}$$

it is obvious that X must be negative since we can take $\beta = -\alpha$. On the other hand β may be chosen to make $am' \sin\theta \sin(\alpha + \beta)$ positive and since b_0 can be made arbitrarily small, (2.4) can be satisfied only if $am' = 0$, i.e. either $m' = 0$ (non-radiating metric) or $a = 0$ (non-rotating metric). However if a/r is sufficiently small, Y is approximately zero and the condition $X < 0$ ensures that $R_{\alpha\beta} V^\alpha V^\beta < 0$ at sufficiently large distances. Therefore $R_{\alpha\beta} V^\alpha V^\beta$ is not necessarily less than zero near the source and so would not necessarily imply positive energy density there.

3. *The Tetrad and Electromagnetic Tensor*

The approximation $(a/r)^2 \ll 1$ is now considered. In this case the two Rainich conditions for a null field (Goodinson and Newing, 1969) are satisfied, $R_\alpha{}^\alpha = 0$ and $R_{\mu\alpha} R^{\alpha\beta} = 0$. Before an electromagnetic field is fitted to this degree of approximation the tetrad of null vectors will be constructed.

In the stated approximation, $\rho^2 = r^2$, $g = \det g_{\alpha\beta} \doteq -r^4 \sin^2\theta$. l_μ is taken to be the null vector $(1, 0, 0, (-3a/2)\sin^2\theta)$ which is Murenbeeld and Trollope's vector ν_μ. The components of the contravariant vector l^μ are then approximately $(0, 1, 0, a/2r^2)$, and $l_\alpha l^\alpha = 0$ to the approximation considered. The vectors m_μ and m^μ are then

$$m_\mu = \frac{1}{\sqrt{2}}\left(0, \frac{a\sin\theta}{2r}, ir, -r\sin\theta\right)$$

$$m^\mu = \frac{1}{\sqrt{2}}\left(\frac{3a\sin\theta}{2r}, \frac{(k-3)\,a\sin\theta}{2r}, -\frac{i}{r}, \frac{1}{r\sin\theta}\right)$$

where $k = 2m/r$.

The self-dual electromagnetic tensor $\mathring{\omega}^{\mu\alpha}$ (Goodinson and Newing, 1969) given by $\mathring{\omega}^{\mu\alpha} = \lambda(l^\mu m^\alpha - l^\alpha m^\mu)$ has the form

$$\mathring{\omega}^{\mu\alpha} = \frac{\lambda}{r\sqrt{2}}\left\{-\frac{3a\sin\theta}{2}\delta_0{}^\mu\,\delta_1{}^\alpha + \delta_1{}^\mu\left(\frac{3a\sin\theta}{2}\delta_0{}^\alpha - i\delta_2{}^\alpha + \frac{1}{\sin\theta}\delta_3{}^\alpha\right)\right.$$
$$\left. + i\delta_2{}^\mu\left(\delta_1{}^\alpha + \frac{a}{2r^2}\delta_3{}^\alpha\right) - \delta_3{}^\mu\left(\frac{ia}{2r^2}\delta_2{}^\alpha + \frac{\delta_1{}^\alpha}{\sin\theta}\right)\right\}. \tag{3.1}$$

A second electromagnetic tensor $\omega^{\mu\alpha}$ can now be introduced by means of a complexion parameter χ (Goodinson & Newing, 1968) in the form $\omega^{\mu\alpha} = e^{i\chi}\mathring{\omega}^{\mu\alpha}$ and the Ricci tensor is then expressed as $R_{\mu\alpha} = -\omega_{\mu\theta}\bar{\omega}^\theta{}_\alpha$. The gravitational field equations require that $\lambda = \sqrt{(-2m')}/r$.

4. *The Maxwell Field Equations*

The Maxwell field equations (Goodinson & Newing, 1968) are

$$\omega^{\mu\alpha}{}_{;\alpha} = \bar{\omega}^{\mu\alpha}{}_{;\alpha} = J^\mu = \frac{1}{\sqrt{-g}}(\sqrt{(-g)}\,.\,e^{i\chi}\mathring{\omega}^{\mu\alpha})_{,\alpha}$$

where J^μ is real (or zero).

When $a = 0$ the Vaidya solution is produced:

$$J^\mu = \frac{K\cot\Lambda}{r^2}\,.\,\delta_1{}^\mu \quad (K = \sqrt{-m'})$$

where

$$\Lambda = \sin^{-1}\{\sin\theta\sin\phi\}.$$

Consider now the case when $a \neq 0$. Since J^μ must be real, the vanishing of the imaginary parts of J^3 and J^1 give

$$\cos\chi\sin\theta = f(u)$$
$$K\sin\chi = F(\theta)$$

where f and F are arbitrary functions of u and θ respectively, subject to

$$\frac{F^2}{K^2} + \frac{f^2}{\sin^2\theta} = 1.$$

The vanishing of the imaginary part of J^0 leads to $\chi_{,1} = 0$ which implies that the real part of J^0 is also zero. If χ is taken to be independent of the

x^3-co-ordinate then the real part of J^2 can be taken to be zero. Thus χ is a function of x^0 and x^2 only.

Introducing new functions $G(u)$ and $H(\theta)$ by the equations

$$\left.\begin{aligned} K\sin\chi\sin\theta &= H(\theta) \\ K\cos\chi &= \frac{G(u)}{\sin\theta} \end{aligned}\right\} \tag{4.1}$$

then

$$G^2 + H^2 = K^2\sin^2\theta. \tag{4.2}$$

Suppose $u = u_1$, $u = u_2$ are two solutions of (4.2) with respect to the same value of θ, then

$$H^2 + G_1{}^2 = K_1{}^2\sin^2\theta$$

$$H + G_2{}^2 = K_2{}^2\sin^2\theta$$

where $G_1 = G(u_1)$ etc. Thus

$$G_1{}^2 - G_2{}^2 = (K_1{}^2 - K_2{}^2)\sin^2\theta.$$

Since u is being varied independently of θ, it can be concluded that $G_1{}^2 - G_2{}^2 = 0$ and $K_1{}^2 = K_2{}^2$ i.e., G and K are constants.

Similarly, varying θ independently of u leads to

$$H_1{}^2 - H_2{}^2 = K^2(\sin^2\theta_1 - \sin^2\theta_2)$$

which being true for all u implies that $K =$ constant, and so $H = K\sin\theta$.

With regard to (4.1), χ can now be taken to be $\pi/2$ with $G = 0$ and $H = K\sin\theta$ where $K =$ constant, i.e., $m' =$ constant.

The expression for J^μ can now be written as

$$J^\mu = \frac{K\cot\theta}{r^2}\left(\delta_1{}^\mu + \frac{a}{2r^2}\delta_3{}^\mu\right). \tag{4.3}$$

Equation (4.3) shows clearly the presence of the extra term $aK\cot\theta/2r^4$ in the $x^3 = \phi$-direction which could be interpreted as a non-radial component of current along the surface of a cone.

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